

# IWASAWA THEORY FOR THE SYMMETRIC SQUARE OF A CM MODULAR FORM AT INERT PRIMES

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**ABSTRACT.** Let  $f$  be a CM modular form and  $p$  an odd prime which is inert in the CM field. We construct two  $p$ -adic  $L$ -functions for the symmetric square of  $f$ , one of which has the same interpolating properties as the one constructed by Delbourgo-Dabrowski, whereas the second one has a similar interpolating properties but corresponds to a different eigenvalue of the Frobenius. The symmetry between these two  $p$ -adic  $L$ -functions allows us to define the plus and minus  $p$ -adic  $L$ -functions à la Pollack. We also define the plus and minus  $p$ -Selmer groups analogous to Kobayashi's Selmer groups. We explain how to relate these two sets of objects via a main conjecture.

## 1. INTRODUCTION

Let  $f$  be a normalised eigen-newform of weight  $k$ , level  $N$  and character  $\epsilon$ . Fix a prime  $p \neq 2$  such that  $p \nmid N$ . In [3] (also in [1] under some additional conditions), even distributions on  $\mathbb{Z}_p^\times$  are constructed to interpolate the  $L$ -values of the symmetric square of  $f$ . More precisely, if the Euler factor of  $L(E, s)$  at  $p$  is given by  $(1 - \alpha_1(p)p^{-s})(1 - \alpha_2(p)p^{-s})$ , then there exists an admissible distribution  $\mu_{\alpha_i(p)^2}$  for  $i = 1, 2$  such that

$$(1) \quad \int_{\mathbb{Z}_p^\times} \theta d\mu_{\alpha_i(p)^2} = \frac{p^{3n(k-1)}}{\alpha_i(p)^{2n} \tau(\theta^{-1})} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(\text{period})}$$

for any non-trivial even Dirichlet character  $\theta$  of conductor  $p^n$  where  $\tau(\theta^{-1})$  denotes the Gauss sum of  $\theta^{-1}$ .

Since the Euler factor of  $L(\text{Sym}^2 f, s)$  at  $p$  is  $(1 - \alpha_1(p)^2 p^{-s})(1 - \alpha_2(p)^2 p^{-s})(1 - \epsilon(p)p^{k-1-s})$ , we expect that there should be a distribution  $\mu_{\epsilon(p)p^{k-1}}$  satisfying interpolating properties similar to (1), but with  $\alpha_i(p)^2$  replaced by  $\epsilon(p)p^{k-1}$ . In this paper, we construct such a distribution for the case when  $f$  is a CM modular form that is non-ordinary at  $p$ . In other words, when the  $L$ -function of  $f$  coincides with that of a Grossencharacter  $\phi$  defined over  $K$  and  $p$  inerts in  $K$ . More precisely, we prove the following theorem in §3 (Theorem 3.20).

**Theorem 1.1.** *If  $f$  is as above, then there exist even admissible distributions  $\mu_{\pm\epsilon(p)p^{k-1}}$  such that*

$$\int_{\mathbb{Z}_p^\times} \theta d\mu_{\pm\epsilon(p)p^{k-1}} = \frac{p^{3n(k-1)}}{(\pm\epsilon(p)p^{k-1})^n \tau(\theta^{-1})} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(\text{period})}.$$

Note that we have  $\alpha_1(p)^2 = \alpha_2(p)^2 = -\epsilon(p)p^{k-1}$  in this case, methods in [3] only produce one distribution, which agrees with  $\mu_{-\epsilon(p)p^{k-1}}$  as given by Theorem 1.1.

The idea of the construction is rather simple. Let  $V_f$  be the  $p$ -adic representation of  $G_{\mathbb{Q}}$  associated to  $f$  as constructed by Deligne in [4]. In order to prove Theorem 1.1, we make use of the following observation. As  $G_{\mathbb{Q}}$ -representations, we have

$$\text{Sym}^2(V_f) \cong V_1 \oplus V_2$$

where  $V_1$  is an one-dimensional representation associated to some Dirichlet character  $\eta$  twisted by a power of the cyclotomic character and  $V_2$  is a two-dimensional representation associated to the Grossencharacter  $\phi^2$ . This implies that the  $L$ -function of  $f$  factorises into

$$L(\text{Sym}^2 f, s) = L(\phi^2, s)L(\eta, s-k+1).$$

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We can therefore make use of an Euler system constructed from elliptic units to interpolate the  $L$ -values of  $\phi^2$  and multiply the resulting distributions with an appropriate twist of the Kubota-Leopoldt  $p$ -adic  $L$ -function associated to  $\eta$ , which interpolates the  $L$ -values of  $\eta$ .

Because of the symmetry between the two distributions, we show that some plus and minus logarithms  $\log^\pm$  of Pollack divide  $\mu_{+\epsilon(p)p^{k-1}} \pm \mu_{-\epsilon(p)p^{k-1}}$ . This allows us to obtain two bounded measures:

**Theorem 1.2.** (*Theorem 3.25*) *Let  $\theta$  be an even Dirichlet character of conductor  $p^n$ . There exist bounded  $p$ -adic measures  $\mu^\pm(\text{Sym}^2(V_f))$  such that the followings hold.*

(a) *If  $n$  is even, then*

$$\int_{\mathbb{Z}_p^\times} \theta \mu^+(\text{Sym}^2(V_f)) = \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\theta(\log^+) \tau(\theta^{-1})^2 \epsilon(p)^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(\text{period})};$$

(b) *If  $n$  is odd, then*

$$\int_{\mathbb{Z}_p^\times} \theta \mu^-(\text{Sym}^2(V_f)) = \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\theta(\log^-) \tau(\theta^{-1})^2 \epsilon(p)^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(\text{period})}.$$

Moreover,  $\mu^\pm(\text{Sym}^2(V_f))$  are uniquely determined by (a) and (b) respectively.

In § 4, we make use of some of the ideas in [6] to show that these measures can be obtained from some appropriate Coleman maps and define the corresponding plus and minus  $p$ -Selmer groups  $\text{Sel}_p^\pm(\text{Sym}^2(V_f))$ . On identifying the measures as elements in some Iwasawa algebra  $\Lambda \otimes \mathbb{Q}$ , we show that the following holds under some appropriate conditions (see Theorem 4.8 for a precise statement).

**Theorem 1.3.** *The Selmer groups  $\text{Sel}_p^\pm(\text{Sym}^2(V_f))$  are  $\Lambda$ -cotorsion and*

$$\text{Char}_{\Lambda \otimes \mathbb{Q}}(\text{Sel}_p^\pm(\text{Sym}^2(V_f))^\vee) = (\mu^\pm(\text{Sym}^2(V_f))).$$

Finally, in the appendix, we explain how some of the linear algebra results that we use to prove the main theorems can be easily generalised to general symmetric powers  $\text{Sym}^m f$  where  $m \geq 2$  is an integer.

## 2. NOTATION

**2.1. Extensions by  $p$  power roots of unity.** Throughout this paper,  $p$  is an odd prime. If  $K$  is a field of characteristic 0, either local or global,  $G_K$  denotes its absolute Galois group,  $\chi$  the  $p$ -cyclotomic character on  $G_K$  and  $\mathcal{O}_K$  the ring of integers of  $K$ . We write  $\iota$  for the complex conjugation in  $G_{\mathbb{Q}}$ .

For an integer  $n \geq 0$ , we write  $K_n$  for the extension  $K(\mu_{p^n})$  where  $\mu_{p^n}$  is the set of  $p^n$ th roots of unity and  $K_\infty$  denotes  $\bigcup_{n \geq 1} K_n$ . When  $K = \mathbb{Q}$ , we write  $k_n = \mathbb{Q}(\mu_{p^n})$  instead. In particular, we write  $\mathbb{Q}_{p,n} = \mathbb{Q}_p(\mu_{p^n})$ . Let  $G_n$  denote the Galois group  $\text{Gal}(\mathbb{Q}_{p,n}/\mathbb{Q}_p)$  for  $0 \leq n \leq \infty$ . Then,  $G_\infty \cong \Delta \times \Gamma$  where  $\Delta = G_1$  is a finite group of order  $p-1$  and  $\Gamma = \text{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_{p,1}) \cong \mathbb{Z}_p$ . We fix a topological generator  $\gamma$  of  $\Gamma$ .

**2.2. Iwasawa algebras and power series.** Given a finite extension  $K$  of  $\mathbb{Q}_p$ ,  $\Lambda_{\mathcal{O}_K}(G_\infty)$  (respectively  $\Lambda_{\mathcal{O}_K}(\Gamma)$ ) denotes the Iwasawa algebra of  $G_\infty$  (respectively  $\Gamma$ ) over  $\mathcal{O}_K$ . We write  $\Lambda_K(G_\infty) = \Lambda_{\mathcal{O}_K}(G_\infty) \otimes K$  and  $\Lambda_K(\Gamma) = \Lambda_{\mathcal{O}_K}(\Gamma) \otimes K$ . If  $M$  is a finitely generated  $\Lambda_{\mathcal{O}_K}(\Gamma)$ -torsion (respectively  $\Lambda_K(\Gamma)$ -torsion) module, we write  $\text{Char}_{\Lambda_{\mathcal{O}_K}(\Gamma)}(M)$  (respectively  $\text{Char}_{\Lambda_K(\Gamma)}(M)$ ) for its characteristic ideal.

Given a module  $M$  over  $\Lambda_{\mathcal{O}_K}(G_\infty)$  (respectively  $\Lambda_K(G_\infty)$ ) and a character  $\delta : \Delta \rightarrow \mathbb{Z}_p^\times$ ,  $M^\delta$  denotes the  $\delta$ -isotypical component of  $M$ . For any  $m \in M$ , we write  $m^\delta$  for the projection of  $m$  into  $M^\delta$ . The Pontryagin dual of  $M$  is written as  $M^\vee$ .

Let  $r \in \mathbb{R}_{\geq 0}$ . We define

$$\mathcal{H}_r = \left\{ \sum_{n \geq 0, \sigma \in \Delta} c_{n,\sigma} \cdot \sigma \cdot X^n \in \mathbb{C}_p[\Delta][[X]] : \sup_n \frac{|c_{n,\sigma}|_p}{n^r} < \infty \ \forall \sigma \in \Delta \right\}$$

where  $|\cdot|_p$  is the  $p$ -adic norm on  $\mathbb{C}_p$  such that  $|p|_p = p^{-1}$ . We write  $\mathcal{H}_\infty = \bigcup_{r \geq 0} \mathcal{H}_r$  and  $\mathcal{H}_r(G_\infty) = \{f(\gamma-1) : f \in \mathcal{H}_r\}$  for  $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ . In other words, the elements of  $\mathcal{H}_r$  (respectively  $\mathcal{H}_r(G_\infty)$ ) are the power series

in  $X$  (respectively  $\gamma - 1$ ) over  $\mathbb{C}_p[\Delta]$  with growth rate  $O(\log_p^r)$ . If  $F, G \in \mathcal{H}_\infty$  or  $\mathcal{H}_\infty(G_\infty)$  are such that  $F = O(G)$  and  $G = O(F)$ , we write  $F \sim G$ .

Given a subfield  $K$  of  $\mathbb{C}_p$ , we write  $\mathcal{H}_{r,K} = \mathcal{H}_r \cap K[\Delta][[X]]$  and similarly for  $\mathcal{H}_{r,K}(G_\infty)$ . In particular,  $\mathcal{H}_{0,K}(G_\infty) = \Lambda_K(G_\infty)$ .

Let  $n \in \mathbb{Z}$ . We define the  $K$ -linear map  $\text{Tw}_n$  from  $\mathcal{H}_{r,K}(G_\infty)$  to itself to be the map that sends  $\sigma$  to  $\chi(\sigma)^n \sigma$  for all  $\sigma \in G_\infty$ . It is clearly bijective (with inverse  $\text{Tw}_{-n}$ ).

**2.3. Crystalline representations.** We write  $\mathbb{B}_{\text{cris}}$  and  $\mathbb{B}_{\text{dR}}$  for the rings of Fontaine and  $\varphi$  for the Frobenius acting on these rings. Recall that there exists an element  $t \in \mathbb{B}_{\text{dR}}$  such that  $\varphi(t) = pt$  and  $g \cdot t = \chi(g)t$  for  $g \in G_{\mathbb{Q}_p}$ .

Let  $V$  be a  $p$ -adic representation of  $G_{\mathbb{Q}_p}$ . We denote the Dieudonné module by  $\mathbb{D}_{\text{cris}}(V) = (\mathbb{B}_{\text{cris}} \otimes V)^{G_{\mathbb{Q}_p}}$ . We say that  $V$  is crystalline if  $V$  has the same  $\mathbb{Q}_p$ -dimension as  $\mathbb{D}_{\text{cris}}(V)$ . Fix such a  $V$ . If  $j \in \mathbb{Z}$ ,  $\text{Fil}^j \mathbb{D}_{\text{cris}}(V)$  denotes the  $j$ th de Rham filtration of  $\mathbb{D}_{\text{cris}}(V)$ .

Let  $T$  be a lattice of  $V$  which is stable under  $G_{\mathbb{Q}_p}$ . Let  $\mathbb{H}_{\text{Iw}}^1(T)$  denote the inverse limit  $\varprojlim \mathbb{H}^1(\mathbb{Q}_{p,n}, T)$  with respect to the corestriction and  $\mathbb{H}_{\text{Iw}}^1(V) = \mathbb{Q} \otimes \mathbb{H}_{\text{Iw}}^1(T)$ . Moreover, if  $V$  arises from the restriction of a  $p$ -adic representation of  $G_{\mathbb{Q}}$  and  $T$  is a lattice stable under  $G_{\mathbb{Q}}$ , we write

$$\mathbb{H}^1(T) = \varprojlim_n H^1(\mathbb{Z}[\mu_{p^n}, 1/p], T) \quad \text{and} \quad \mathbb{H}^1(V) = \mathbb{Q} \otimes \mathbb{H}^1(T).$$

We have localisation maps

$$\text{loc} : \mathbb{H}^1(T) \rightarrow \mathbb{H}_{\text{Iw}}^1(T) \quad \text{and} \quad \text{loc} : \mathbb{H}^1(V) \rightarrow \mathbb{H}_{\text{Iw}}^1(V).$$

If  $F$  is a number field, we define the  $p$ -Selmer group of  $T$  over  $F$  to be

$$\text{Sel}_p(T/F) = \ker \left( H^1(K, T \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \prod_v \frac{H^1(F_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{H_f^1(F_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \right)$$

where  $v$  runs through the places of  $F$ .

Let  $V(j)$  denote the  $j$ th Tate twist of  $V$ , i.e.  $V(j) = V \otimes \mathbb{Q}_p e_j$  where  $G_{\mathbb{Q}_p}$  acts on  $e_j$  via  $\chi^j$ . We have

$$\mathbb{D}_{\text{cris}}(V(j)) = t^{-j} \mathbb{D}_{\text{cris}}(V) \otimes e_j.$$

For any  $v \in \mathbb{D}_{\text{cris}}(V)$ ,  $v_j = v \otimes t^{-j} e_j$  denotes its image in  $\mathbb{D}_{\text{cris}}(V(j))$ . We write  $\text{Tw}_j : \mathbb{H}_{\text{Iw}}^1(V) \rightarrow \mathbb{H}_{\text{Iw}}^1(V(j))$  for the isomorphism defined in [9, § A.4], which depends on a choice of primitive  $p$ -power roots of unity.

Finally, we write

$$\exp : \mathbb{Q}_{p,n} \otimes \mathbb{D}_{\text{cris}}(V) \rightarrow H^1(\mathbb{Q}_{p,n}, V) \quad \text{and} \quad \exp^* : H^1(\mathbb{Q}_{p,n}, V) \rightarrow \mathbb{Q}_{p,n} \otimes \text{Fil}^0 \mathbb{D}_{\text{cris}}(V)$$

for Bloch-Kato's exponential and dual exponential respectively.

**2.4. Imaginary quadratic fields.** Let  $K$  be an imaginary quadratic field with ring of integers  $\mathcal{O}$  and idele class group  $C_K$ . We write  $\varepsilon_K$  for the quadratic character associated to  $K$ , i.e. the character on  $G_{\mathbb{Q}}$  which sends  $\sigma$  to 1 if  $\sigma \in G_K$  and to  $-1$  otherwise.

A Grossencharacter of  $K$  is simply a continuous homomorphism  $\phi : C_K \rightarrow \mathbb{C}^\times$  with complex  $L$ -function

$$L(\phi, s) = \prod_v (1 - \phi(v)N(v)^{-s})^{-1}$$

where the product runs through the finite places  $v$  of  $K$  at which  $\phi$  is unramified,  $\phi(v)$  is the image of the uniformiser of  $K_v$  under  $\phi$  and  $N(v)$  is the norm of  $v$ . Let  $\mathfrak{f}$  be the conductor of  $\phi$ . We say that  $\eta$  is of type  $(m, n)$  where  $m, n \in \mathbb{Z}$  if the restriction of  $\eta$  to the archimedean part  $\mathbb{C}^\times$  of  $C_K$  is of the form  $z \mapsto z^m \bar{z}^n$ .

We write  $\mathcal{K} = \cup K(p^n \mathfrak{f})$  where  $K(\mathfrak{a})$  denotes the ray class field of  $K$  modulo  $\mathfrak{a}$  if  $\mathfrak{a}$  is an ideal of  $\mathcal{O}$ .

If  $T$  is a  $\mathbb{Z}_p$ -representation of  $G_K$ , we write

$$\mathbb{H}_{p^\infty \mathfrak{f}}^1(T) = \varprojlim_{K'} H^1(\mathcal{O}_{K'}[1/p], T) \quad \text{and} \quad \mathbb{H}_{p^\infty \mathfrak{f}}^1(\mathbb{Q} \otimes_{\mathbb{Z}_p} T) = \mathbb{H}_{p^\infty \mathfrak{f}}^1(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}$$

where  $K'$  ranges over all finite extensions of  $K$  contained in  $K(p^\infty \mathfrak{f})$ .

**2.5. Modular forms.** Let  $f = \sum a_n q^n$  be a normalised eigen-newform of weight  $k \geq 2$ , level  $N$  and character  $\epsilon$ . We assume that  $f$  is a CM modular form, i.e.  $L(f, s) = L(\phi, s)$  for some Grossencharacter  $\phi$  of an imaginary quadratic field  $K$  with conductor  $\mathfrak{f}$ . Then,  $\phi$  is of type  $(-k+1, 0)$ . Moreover,  $p$  inert in  $K$  if and only if  $f$  is non-ordinary at  $p$ . In this case,  $a_p$  is always 0. Throughout, we fix such a  $p$  with  $p \neq 2$ .

The coefficient field  $F_f$  of  $f$  is contained in the field of definition of  $\phi$ . We write  $E$  for the completion of this field at a fixed prime above  $p$ .

We write  $V_f$  for the 2-dimensional  $E$ -linear representation of  $G_{\mathbb{Q}}$  associated to  $f$  from [4], so we have a homomorphism

$$\rho_f : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(V_f).$$

Throughout the paper, we assume that the following hypothesis holds.

**Hypothesis 2.1.** *If  $\epsilon$  and  $K$  are as above, then  $\varepsilon_K \neq \epsilon$ .*

### 3. $p$ -ADIC $L$ -FUNCTIONS

**3.1. Grossencharacters over  $K$ .** We first review some results on Grossencharacters. Let  $\eta$  be a Grossencharacter on  $G_K$  of conductor  $\mathfrak{f}$ . We fix a finite extension  $E$  of  $\mathbb{Q}_p$  such that  $E$  contains the image of  $\eta$ . We write  $V(\eta)$  for the one-dimensional  $E$ -linear representation of  $G_K$ . It is a representation that factors through  $\mathrm{Gal}(\mathcal{K}/K)$ . For an ideal  $\mathfrak{a}$  of  $\mathcal{O}$  which is prime to  $p\mathfrak{f}$ , the Artin symbol  $(\mathfrak{a}, \mathcal{K}/K) \in \mathrm{Gal}(\mathcal{K}/K)$  acts on  $V(\eta)$  as the multiplication by  $\eta(\mathfrak{a})^{-1}$ . We write  $\tilde{\eta} : G_K \rightarrow E^\times$  for the corresponding character.

We write  $\tilde{V}_\eta = \mathrm{Ind}_K^{\mathbb{Q}}(V(\eta))$ . The canonical homomorphism  $K \otimes \mathbb{Q}(\zeta_{p^\infty}) \rightarrow K(p^\infty \mathfrak{f})$  induces a map

$$\mathrm{Ind} : \mathbb{H}_{p^\infty \mathfrak{f}}^1(V(\eta)) \rightarrow \mathbb{H}^1(\tilde{V}_\eta).$$

Let  $\gamma$  be a non-zero element of  $V(\eta)$ . By [5, §15.5], a system of norm compatible elliptic units in  $K(p^n \mathfrak{f})$  defines an element  $z_{p^\infty \mathfrak{f}} \in \mathbb{H}_{p^\infty \mathfrak{f}}^1(\mathbb{Z}_p(1))$ . We write the image of  $z_{p^\infty \mathfrak{f}}$  under the composition

$$\mathbb{H}_{p^\infty \mathfrak{f}}^1(\mathbb{Z}_p(1)) \xrightarrow{\gamma} \mathbb{H}_{p^\infty \mathfrak{f}}^1(V(\eta)(1)) \xrightarrow{\mathrm{Ind}} \mathbb{H}^1(\tilde{V}_\eta(1)) \xrightarrow{\mathrm{loc}} \mathbb{H}_{\mathrm{Iw}}^1(\tilde{V}_\eta(1)) \xrightarrow{\mathrm{Tw}_{-1}} \mathbb{H}_{\mathrm{Iw}}^1(\tilde{V}_\eta)$$

as  $z_\gamma(\eta) = z(\eta)$  and its projection into  $H^1(\mathbb{Q}_{p,n}, \tilde{V}_\eta(j))$  is denoted by  $z_{j,n}(\eta)$ .

Note that the eigenvalues of  $\iota$  on  $\tilde{V}_\eta$  are  $\pm 1$ , each with multiplicity 1. If  $v \in \tilde{V}_\eta$ , we write  $v^\pm$  for the projection of  $v$  into the  $\pm 1$ -eigenspace.

**Proposition 3.1.** *Let  $\eta$  be a Grossencharacter over  $K$  of type  $(-r, 0)$  with  $r \geq 1$ . Let  $\theta$  be a character on  $G_n$  and write*

$$\begin{aligned} \kappa_\theta : \mathbb{Q}_{p,n} \otimes \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(\tilde{V}_\eta(1)) &\rightarrow \mathbb{C} \otimes \tilde{V}_\eta(1) \\ x \otimes y &\mapsto \sum_{\sigma \in G_n} \theta(\sigma) \sigma(x) \mathrm{per}(y) \end{aligned}$$

where  $\mathrm{per}$  is the period map associated to  $\eta$  as defined in [5, §15.8]. Then, we have

$$\kappa_\theta \circ \exp^*(z_{1,n}(\eta)) = L_{\{p\}}(\bar{\eta}\theta, r) \cdot (\gamma')^\pm$$

where  $\pm = \theta(-1)$  and  $\gamma'$  denotes the image of  $\gamma$  in  $\tilde{V}_\eta$ .

*Proof.* [5, §15.12]. □

**3.2. The symmetric square of a CM modular form.** Let  $f$  be a modular form as in §2.5. By comparing the eigenvalues of Frobenii, we see that the representation  $V_f$  is isomorphic to  $\tilde{V}_\phi = \mathrm{Ind}_K^{\mathbb{Q}} V(\phi)$ . Therefore,  $V_f$  admits a basis  $x, y$  such that for  $\sigma \in G_{\mathbb{Q}}$ , the matrix of  $\rho_f(\sigma)$  with respect to this basis is given by

$$(2) \quad \rho_f(\sigma) = \begin{pmatrix} \tilde{\phi}(\sigma) & 0 \\ 0 & \tilde{\phi}(\iota\sigma\iota) \end{pmatrix}$$

if  $\sigma \in G_K$ . Otherwise,

$$(3) \quad \rho_f(\sigma) = \begin{pmatrix} 0 & \tilde{\phi}(\iota\sigma'\iota) \\ \tilde{\phi}(\sigma') & 0 \end{pmatrix}$$

where  $\sigma = \iota\sigma'$  with  $\sigma' \in G_K$ .

**Lemma 3.2.** *The determinant of  $\rho_f$  is given by*

$$\det(\rho_f)(\sigma) = \begin{cases} \tilde{\phi}(\sigma)\tilde{\phi}(\iota\sigma\iota) & \text{if } \sigma \in G_K \\ -\tilde{\phi}(\sigma')\tilde{\phi}(\iota\sigma'\iota) & \text{if } \sigma = \iota\sigma' \text{ where } \sigma' \in G_K. \end{cases}$$

*Proof.* This is immediate from (2) and (3).  $\square$

**Proposition 3.3.** *As a  $G_{\mathbb{Q}}$ -representation,  $\text{Sym}^2(V_f)$  decomposes into*

$$\text{Sym}^2(V_f) \cong V_1 \oplus V_2$$

where  $\rho_i : G_{\mathbb{Q}} \rightarrow \text{GL}(V_i)$  is an  $i$ -dimensional representation of  $G_{\mathbb{Q}}$  for  $i = 1, 2$ . Moreover,

$$(4) \quad \rho_1 \cong \varepsilon_K \cdot \det(\rho_f) = \varepsilon_K \cdot \epsilon \cdot \chi^{k-1},$$

$$(5) \quad \rho_2 \cong \tilde{V}_{\phi^2}.$$

*Proof.* It is clear that  $x \otimes x$ ,  $y \otimes y$ ,  $x \otimes y + y \otimes x$  form a basis of  $\text{Sym}^2(V_f)$ . By formulae (2) and (3),  $\sigma \cdot (x \otimes y + y \otimes x)$  is a multiple of  $x \otimes y + y \otimes x$  for any  $\sigma \in G_{\mathbb{Q}}$ . Hence, it gives an one-dimensional sub-representation  $V_1$  of  $\text{Sym}^2(V_f)$ . More explicitly, we have

$$\sigma \cdot (x \otimes y + y \otimes x) = \begin{cases} \tilde{\phi}(\sigma)\tilde{\phi}(\iota\sigma\iota)(x \otimes y + y \otimes x) & \text{if } \sigma \in G_K \\ \tilde{\phi}(\sigma')\tilde{\phi}(\iota\sigma'\iota)(x \otimes y + y \otimes x) & \text{if } \sigma = \iota\sigma' \text{ where } \sigma' \in G_K. \end{cases}$$

Therefore, we deduce (4) from Lemma 3.2.

It is also clear that  $x \otimes x$ ,  $y \otimes y$  form a basis of a 2-dimensional representation  $\rho_2 : G_{\mathbb{Q}} \rightarrow \text{GL}(V_2)$ . With respect to this basis,

$$\rho_2(\sigma) = \begin{pmatrix} \tilde{\phi}^2(\sigma) & 0 \\ 0 & \tilde{\phi}^2(\iota\sigma\iota) \end{pmatrix}$$

if  $\sigma \in G_K$ . Otherwise, if  $\sigma = \iota\sigma'$  where  $\sigma' \in G_K$ , then

$$\rho_2(\sigma) = \begin{pmatrix} 0 & \tilde{\phi}^2(\iota\sigma'\iota) \\ \tilde{\phi}^2(\sigma') & 0 \end{pmatrix}.$$

Therefore,  $V_2 \cong \text{Ind}_K^{\mathbb{Q}} V(\phi^2)$  as required.  $\square$

**Corollary 3.4.** *The complex  $L$  function admits a factorisation*

$$L(\text{Sym}^2 f, s) = L(\phi^2, s) L(\varepsilon_K \cdot \epsilon, s - k + 1).$$

*Proof.* The  $L$ -function of  $\text{Sym}^2 f$  only have non-trivial Euler factors at  $q \nmid N$ . The Euler factors on the two sides of the equation at  $q$  agree by Proposition 3.3, so we are done.  $\square$

**3.3. The symmetric square as a  $G_{\mathbb{Q}_p}$ -representation.** We study the representation  $\text{Sym}^2(V_f)$  restricted to  $G_{\mathbb{Q}_p}$ . More specifically, we study  $\mathbb{D}_{\text{cris}}(\text{Sym}^2 V_f)$ .

**Lemma 3.5.** *As  $G_{\mathbb{Q}_p}$ -representations, both  $V_1$  and  $V_2$  are crystalline.*

*Proof.* The functor  $\mathbb{D}_{\text{cris}}$  is compatible with taking direct sums, so we can identify  $\mathbb{D}_{\text{cris}}(V_i)$  as a filtered sub- $\varphi$ -module of  $\mathbb{D}_{\text{cris}}(V_f)$  for  $i = 1, 2$ . That is,

$$(6) \quad \mathbb{D}_{\text{cris}}(\text{Sym}^2(V_f)) \cong \mathbb{D}_{\text{cris}}(V_1) \oplus \mathbb{D}_{\text{cris}}(V_2).$$

Since  $\text{Sym}^2(V_f)$  is crystalline, so  $\mathbb{D}_{\text{cris}}(\text{Sym}^2(V_f))$  is of dimension 3 over  $E$ . Hence,  $\mathbb{D}_{\text{cris}}(V_i)$  must have dimension  $i$  and  $V_i$  is crystalline for  $i = 1, 2$ .  $\square$

We now give explicit descriptions of  $\mathbb{D}_{\text{cris}}(V_1)$  and  $\mathbb{D}_{\text{cris}}(V_2)$ .

Recall that  $\mathbb{D}_{\text{cris}}(V_f)$  is a 2-dimensional  $E$ -vector space with Hodge-Tate weights 0 and  $1 - k$ . Moreover, the de Rham filtration is given by

$$(7) \quad \text{Fil}^i \mathbb{D}_{\text{cris}}(V_f) = \begin{cases} E\omega \oplus E\varphi(\omega) & \text{if } i \leq 0 \\ E\omega & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i \geq k \end{cases}$$

for some  $\omega \neq 0$ . The action of  $\varphi$  on  $\mathbb{D}_{\text{cris}}(V_f)$  satisfies  $\varphi^2 = -\epsilon(p)p^{k-1}$ . Therefore,

$$(8) \quad \text{Fil}^i \mathbb{D}_{\text{cris}}(\text{Sym}^2(V_f)) = \begin{cases} \mathbb{D}_{\text{cris}}(\text{Sym}^2(V_f)) & \text{if } i \leq 0 \\ E(\omega \otimes \omega) \oplus E(\varphi(\omega) \otimes \omega + \omega \otimes \varphi(\omega)) & \text{if } 1 \leq i \leq k-1 \\ E(\omega \otimes \omega) & \text{if } k \leq i \leq 2k-2 \\ 0 & \text{if } i \geq 2k-1 \end{cases}$$

Since  $\varphi^2(\omega) = -\epsilon(p)p^{k-1}\omega$ , we have

$$\varphi(\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega) = -\epsilon(p)p^{k-1}(\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega).$$

In particular,  $\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega$  is an eigenvector of  $\varphi$ . Therefore, we have a decomposition of filtered  $\varphi$ -modules

$$\mathbb{D}_{\text{cris}}(\text{Sym}^2(V_f)) = (E(\omega \otimes \omega) \oplus E(\varphi(\omega) \otimes \varphi(\omega))) \oplus (E(\omega \otimes \varphi(\omega) + \varphi(\omega) \otimes \omega)).$$

**Proposition 3.6.** *As filtered  $\varphi$ -modules, we have*

$$\begin{aligned} \mathbb{D}_{\text{cris}}(V_1) &= E(\varphi(\omega) \otimes \omega + \omega \otimes \varphi(\omega)), \\ \mathbb{D}_{\text{cris}}(V_2) &= E(\omega \otimes \omega) \oplus E(\varphi(\omega) \otimes \varphi(\omega)). \end{aligned}$$

*Proof.* By (4),  $\rho_1 = \varepsilon_K \cdot \epsilon \cdot \chi^{k-1}$ . Since  $p$  is inert in  $K$ ,  $\varepsilon_K(p) = -1$ . The Hodge-Tate weight of  $V_1$  is therefore  $1 - k$  and  $\varphi$  acts on  $\mathbb{D}_{\text{cris}}(V_1)$  as multiplication by  $-\epsilon(p)p^{k-1}$ . This proves the first equality. The second equality is then automatic by (6).  $\square$

**Remark 3.7.** *Such a decomposition of  $G_{\mathbb{Q}_p}$ -representations is in fact possible for  $f$  without CM (see [10, §2.2]).*

**Corollary 3.8.** *The eigenvalues of  $\varphi$  on  $\mathbb{D}_{\text{cris}}(V_2)$  are  $\pm\epsilon(p)p^{k-1}$ .*

*Proof.* By Proposition 3.6, the matrix of  $\varphi$  with respect to the basis  $\omega \otimes \omega, \varphi(\omega) \otimes \varphi(\omega)$  is

$$\begin{pmatrix} 0 & \epsilon(p)^2 p^{2k-2} \\ 1 & 0 \end{pmatrix},$$

hence the result.  $\square$

**Corollary 3.9.** *The Hodge-Tate weights of  $V_2$  are 0 and  $2 - 2k$ .*

*Proof.* This follows from (8) and Proposition 3.6.  $\square$

**3.4. The Perrin-Riou pairing.** By Corollary 3.8, the slope of  $\varphi$  on  $\mathbb{D}_{\text{cris}}(V_2)$  is  $k - 1$ . Hence, by Corollary 3.9, given any  $v \in \mathbb{D}_{\text{cris}}(V_2)$ , we have the Perrin-Riou pairing

$$\mathcal{L}_v : \mathbb{H}_{\text{IW}}^1(V_2^*) \rightarrow \mathcal{H}_{k-1, E}(G_{\infty})$$

which satisfies the following properties.

**Proposition 3.10.** *For an integer  $r \geq 0$ , we have*

$$\chi^r(\mathcal{L}_v(\mathbf{z})) = r! \left[ \left( 1 - \frac{\varphi^{-1}}{p} \right) (1 - \varphi)^{-1}(v_{r+1}), \exp^*(z_{-r,0}) \right]_0.$$

Let  $\theta$  be a character of  $G_n$  which does not factor through  $G_{n-1}$  with  $n \geq 1$ , then

$$\chi^r \theta(\mathcal{L}_v(\mathbf{z})) = \frac{r!}{\tau(\theta^{-1})} \sum_{\sigma \in G_n} \theta^{-1}(\sigma) [\varphi^{-n}(v_{r+1}), \exp^*(z_{-r,n}^\sigma)]_n$$

where  $[\cdot]_n$  is the pairing

$$[\cdot]_n : H^1(\mathbb{Q}_{p,n}, V_2(r+1)) \times H^1(\mathbb{Q}_{p,n}, V_2^*(-r)) \rightarrow H^2(\mathbb{Q}_{p,n}, E(1)) \cong E,$$

$z_{-r,n}$  denotes the projection of  $\text{Tw}_{-r}(z)$  into  $H^1(\mathbb{Q}_{p,n}, V_2^*(-r))$  and  $\tau(\theta^{-1})$  denotes the Gauss sum of  $\theta^{-1}$ .

*Proof.* See [6, §3.2].  $\square$

**Remark 3.11.** The assumption on the eigenvalues of  $\varphi$  made in [6] are not necessary for our purposes here because the Perrin-Riou pairings can be defined by applying  $1 - \varphi$  to the  $(\varphi, G_\infty)$ -module of  $V_2^*$  (see [7] and [5, §16.4]).

We fix a non-zero element  $\bar{\omega} \in \text{Fil}^{-1} \mathbb{D}_{\text{cris}}(V_2^*(1))$  and write

$$\text{per}(\bar{\omega}) = \Omega_+(\gamma')^+ + \Omega_-(\gamma')^-$$

where  $\Omega_\pm \in \mathbb{C}^\times$  and  $\gamma'$  is as in the statement of Proposition 3.1 for some fixed  $\gamma$ .

**Definition 3.12.** Under the choices made above, we define  $v^\pm \in \mathbb{D}_{\text{cris}}(V_2)$  by

$$v^\pm = \frac{1}{[\varphi(\omega) \otimes \varphi(\omega), \bar{\omega}]} \left( \pm \epsilon(p) p^{k-1} \omega \otimes \omega + \varphi(\omega) \otimes \varphi(\omega) \right).$$

**Lemma 3.13.** The elements  $v^\pm$  satisfy:

- (a) Both  $v^\pm$  are eigenvalues of  $\varphi$  with  $\varphi(v^\pm) = \pm \epsilon(p) p^{k-1} v^\pm$ ;
- (b) For any  $x \in \text{Fil}^0 \mathbb{D}_{\text{cris}}(V_2^*(-r))$  and an integer  $r$  such that  $0 \leq r \leq 2k-3$ , we have

$$[v_{r+1}^+, x] = [v_{r+1}^-, x]$$

where  $[\cdot]$  denotes the pairing

$$[\cdot] : \mathbb{D}_{\text{cris}}(V_2(r+1)) \times \mathbb{D}_{\text{cris}}(V_2^*(-r)) \rightarrow \mathbb{D}_{\text{cris}}(E(1)) = E \cdot t^{-1} e_1.$$

*Proof.* (a) is easy to check using the matrix given in the proof of Corollary 3.8 (or by direct calculations).

By Corollary 3.9, the Hodge-Tate weights of  $V_2^*$  are 0 and  $2k-2$ . Hence,  $\text{Fil}^0 \mathbb{D}_{\text{cris}}(V_2^*(-r))$  is one-dimensional with basis  $\bar{\omega}_{-r-1}$  for  $0 \leq r \leq 2k-3$ . Since  $(\omega \otimes \omega)_{r+1} \in \text{Fil}^0 \mathbb{D}_{\text{cris}}(V_2(r+1))$ , we have  $[(\omega \otimes \omega)_{r+1}, \bar{\omega}_{-r-1}] = 0$ . Hence,

$$[v_{r+1}^+, \bar{\omega}_{-r-1}] = [v_{r+1}^-, \bar{\omega}_{-r-1}] = 1,$$

which implies (b).  $\square$

Note that  $V_2^* \cong \tilde{V}_{\bar{\phi}^2}(2k-2)$ . This enables us to make the following definition of  $p$ -adic  $L$ -functions associated to  $\phi^2$ .

**Definition 3.14.** On taking  $\eta = \bar{\phi}^2$  in §3.1, we define

$$L_{\pm \epsilon(p) p^{k-1}}(\phi^2) = \mathcal{L}_{v^\pm}(\text{Tw}_{2k-2}(z(\bar{\phi}^2))) \in \mathcal{H}_{k-1, E}(G_\infty).$$

**Lemma 3.15.** Let  $\theta$  be a character of  $G_n$  which does not factor through  $G_{n-1}$  with  $n \geq 1$  and write  $\delta = \theta(-1)$ , then

$$\chi^{2k-3} \theta(L_\alpha(\phi^2)) = \frac{(2k-3)! p^{(2k-2)n}}{\tau(\theta^{-1}) \alpha^n} \times \frac{L(\phi^2 \theta^{-1}, 2k-2)}{\Omega_\delta}$$

where  $\alpha = \pm \epsilon(p) p^{k-1}$ .

*Proof.* We have

$$\begin{aligned}
& \chi^{2k-3} \theta \left( L_{\pm \epsilon(p)p^{k-1}}(\phi^2) \right) \\
&= \chi^{2k-3} \theta \left( \mathcal{L}_{v^\pm} \left( \text{Tw}_{2k-2} \left( z(\bar{\phi}^2) \right) \right) \right) \\
&= \frac{(2k-3)!}{\tau(\theta^{-1})} \sum_{\sigma \in G_n} \theta^{-1}(\sigma) \left[ \varphi^{-n}(v_{2k-2}^\pm), \exp^*(z_{1,n}(\bar{\phi}^2)^\sigma) \right]_n \\
&= \frac{(2k-3)!}{\tau(\theta^{-1})} \left[ \left( \pm \epsilon(p)p^{k-1} \times p^{-2k+2} \right)^{-n} v_{2k-2}^\pm, \sum_{\sigma \in G_n} \theta^{-1}(\sigma) \exp^*(z_{1,n}(\bar{\phi}^2)^\sigma) \right]_n \\
&= \frac{(2k-3)! p^{(2k-2)n}}{\tau(\theta^{-1}) (\pm \epsilon(p)p^{k-1})^n} \times \frac{L(\phi^2 \theta^{-1}, 2k-2)}{\Omega_\delta}
\end{aligned}$$

where the second equality follows from Proposition 3.10, the third follows from Lemma 3.13(a) and the last equality is a consequence of Proposition 3.1 and the fact that  $p$  divides the conductor of  $\theta$ .  $\square$

**Lemma 3.16.** *We have*

$$\chi^{2k-3} \left( L_{\pm \epsilon(p)p^{k-1}}(\phi^2) \right) = \left( 1 - p^{-1} + (1 - \epsilon(p)^{-2} p^{2k-3})(\pm \epsilon(p)p^{1-k}) \right) \times \frac{L(\phi^2, 2k-2)}{\Omega_+}.$$

*Proof.* Since  $\varphi^2 = \epsilon(p)^2 p^{2-2k}$  on  $\mathbb{D}_{\text{cris}}(V_2(2k-2))$ , we have

$$\begin{aligned}
& \left( 1 - \frac{\varphi^{-1}}{p} \right) (1 - \varphi)^{-1} \\
&= (1 - \epsilon(p)^{-2} p^{2k-3} \varphi) \frac{1 + \varphi}{1 - \epsilon(p)^2 p^{2-2k}} \\
&= \frac{1 - p^{-1} + (1 - \epsilon(p)^{-2} p^{2k-3}) \varphi}{1 - \epsilon(p)^2 p^{2-2k}}.
\end{aligned}$$

Therefore, similarly to the proof of Lemma 3.15, we have

$$\begin{aligned}
& \chi^{2k-3} \left( L_{\pm \epsilon(p)p^{k-1}}(\phi^2) \right) \\
&= \chi^{2k-3} \left( \mathcal{L}_{v^\pm} \left( \text{Tw}_{2k-2} \left( z(\bar{\phi}^2) \right) \right) \right) \\
&= (2k-3)! \left[ \frac{1 - p^{-1} + (1 - \epsilon(p)^{-2} p^{2k-3}) \varphi}{1 - \epsilon(p)^2 p^{2-2k}} (v_{2k-2}^\pm), \exp^*(z_{1,0}(\bar{\phi}^2)) \right]_0 \\
&= (2k-3)! \left[ \frac{1 - p^{-1} + (1 - \epsilon(p)^{-2} p^{2k-3})(\pm \epsilon(p)p^{1-k})}{1 - \epsilon(p)^2 p^{2-2k}} \cdot v_{2k-2}^\pm, \exp^*(z_{1,0}(\bar{\phi}^2)) \right]_0 \\
&= \frac{1 - p^{-1} + (1 - \epsilon(p)^{-2} p^{2k-3})(\pm \epsilon(p)p^{1-k})}{1 - \epsilon(p)^2 p^{2-2k}} \times \frac{L_{\{p\}}(\phi^2, 2k-2)}{\Omega_+} \\
&= \left( 1 - p^{-1} + (1 - \epsilon(p)^{-2} p^{2k-3})(\pm \epsilon(p)p^{1-k}) \right) \times \frac{L(\phi^2, 2k-2)}{\Omega_+}.
\end{aligned}$$

$\square$

**Remark 3.17.** *Consider the  $p$ -adic  $L$ -function  $L_{+\epsilon(p)p^{k-1}}(\phi^2)$ . The first factor on the right-hand side of the equation in the statement of Lemma 3.16 vanishes if and only if  $k = 2$  and  $\epsilon(p) = 1$  (e.g. when  $f$  corresponds to an elliptic curve over  $\mathbb{Q}$ ). This recovers the trivial zero result in [10].*



**3.5.  $p$ -adic  $L$ -functions of the symmetric square.** Let us first recall the following result of Kubota and Leopoldt.

**Theorem 3.18.** *If  $\eta$  is a non-trivial Dirichlet character of conductor prime to  $p$ , there exists a bounded  $p$ -adic measure  $L_p(\eta) \in \mathcal{H}_{0,E}(G_\infty)$  where  $E$  is some finite extension of  $\mathbb{Q}_p$  which contains the image of  $\eta$  such that*

$$\begin{aligned}\chi^r \theta(L_p(\eta)) &= \frac{(r+1)! p^{n(r+1)}}{(2\pi i)^{r+1} \tau(\theta^{-1})} \times L(\eta \theta^{-1}, r+1); \\ \chi^r(L_p(\eta)) &= \frac{(r+1)!}{(2\pi i)^{r+1}} L(\eta, r+1).\end{aligned}$$

for any integer  $r \geq 0$  and Dirichlet character  $\theta$  of conductor  $p^n$  such that  $\chi^{r+1} \theta(-1) = \eta(-1)$ .

Since we assume that Hypothesis 2.1 holds, we may take  $\eta = \varepsilon_K \cdot \epsilon$  in Theorem 3.18. This enables us to give the following definition.

**Definition 3.19.** For  $\alpha = \pm \epsilon(p) p^{k-1}$  we define

$$L_\alpha(\text{Sym}^2(V_f)) = L_\alpha(\phi^2) \times \text{Tw}_{-k+1}(L_p(\varepsilon_K \cdot \epsilon)).$$

For the rest of this section, unless otherwise stated,  $\theta$  denotes an even character on  $G_n$  which does not factor through  $G_{n-1}$  with  $n \geq 1$ .

**Theorem 3.20.** *Both  $L_{\pm \epsilon(p) p^{k-1}}(\text{Sym}^2(V_f))$  lie inside  $\mathcal{H}_{k-1,E}(G_\infty)$  and admit the following interpolating properties:*

$$\begin{aligned}\chi^{2k-3} \theta(L_\alpha(\text{Sym}^2(V_f))) &= \frac{(2k-3)!(k-1)! p^{3n(k-1)}}{\tau(\theta^{-1})^2 \alpha^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1} \Omega_+}; \\ \chi^{2k-3} (L_\alpha(\text{Sym}^2(V_f))) &= (2k-3)!(k-1)! \left(1 - \frac{1}{p} + \alpha \left(p^{-2k+2} - \frac{1}{p\epsilon(p)^2}\right)\right) \times \frac{L(\text{Sym}^2 f, 2k-2)}{(2\pi i)^{k-1} \Omega_+}\end{aligned}$$

where  $\alpha = \pm \epsilon(p) p^{k-1}$ .

*Proof.* By definition,  $L_\alpha(\phi^2) \in \mathcal{H}_{k-1,E}(G_\infty)$  and  $L_p(\varepsilon_K \cdot \epsilon) \in \mathcal{H}_{0,E}(G_\infty)$  which implies the first part of the theorem.

Since  $\det(V_f) = \epsilon \chi^{k-1}$  and  $\rho_f$  is odd, we have  $\epsilon \chi^{k-1}(-1) = -1$ . But  $\varepsilon_K(-1) = -1$  and  $\theta(-1) = 1$ , so  $\chi^{k-1} \theta(-1) = \varepsilon_K \epsilon(-1)$  and we can apply Theorem 3.18 and Lemma 3.15 as follows:

$$\begin{aligned}&\chi^{2k-3} \theta(L_\alpha(\text{Sym}^2(V_f))) \\ &= \chi^{2k-3} \theta(L_\alpha(\phi^2)) \times \chi^{k-2} \theta(L_p(\varepsilon_K \cdot \epsilon)) \\ &= \frac{(2k-3)! p^{(2k-2)n}}{\tau(\theta^{-1}) \alpha^n} \times \frac{L(\phi^2 \theta, 2k-2)}{\Omega_+} \times \frac{(k-1)! p^{n(k-1)}}{(2\pi i)^{k-1} \tau(\theta^{-1})} \times L(\varepsilon_K \cdot \epsilon \cdot \theta^{-1}, k-1) \\ &= \frac{(2k-3)!(k-1)! p^{3n(k-1)}}{\tau(\theta^{-1})^2 \alpha^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1} \Omega_+},\end{aligned}$$

where the last equality follows from Corollary 3.4. This gives the first interpolating formula and the second one can be deduced in the same way.  $\square$

**Lemma 3.21.** *Let  $\eta$  be an even character on  $\Delta$ , then  $L_{\pm \epsilon(p) p^{k-1}}^\eta(\text{Sym}^2(V_f)) \neq 0$ .*

*Proof.* We have  $L(\text{Sym}^2(V_f), \eta, 2k-2) \neq 0$  because the critical strip of  $\text{Sym}^2(V_f)$  is  $k-1 < \text{Re}(s) < k$ . Therefore, we are done by the interpolating properties given by Theorem 3.20.  $\square$

**3.6. Pollack's plus and minus splittings.** As in [11], we define

$$\begin{aligned}\log^+(\gamma) &= \prod_{r=0}^{2k-3} \prod_{n=1}^{\infty} \frac{\Phi_{2n}(\chi(\gamma)^{-r}\gamma)}{p}, \\ \log^-(\gamma) &= \prod_{r=0}^{2k-3} \prod_{n=1}^{\infty} \frac{\Phi_{2n-1}(\chi(\gamma)^{-r}\gamma)}{p},\end{aligned}$$

where  $\Phi_m$  denotes the  $p^m$ th cyclotomic polynomial. Then,  $\log^\pm(\gamma) \sim \log^{k-1}$ .

**Lemma 3.22.** *For an integer  $r$  such that  $0 \leq r \leq 2k-3$  and a character  $\theta$  of  $G_n$  which does not factor through  $G_{n-1}$  with  $n \geq 1$ , then*

$$\chi^r \theta \left( L_{+\epsilon(p)p^{k-1}}(\phi^2) \right) = (-1)^n \chi^r \theta \left( L_{-\epsilon(p)p^{k-1}}(\phi^2) \right).$$

*Proof.* This follows from the same calculations as in the proof of Lemma 3.15 thanks to Lemma 3.13(b).  $\square$

**Corollary 3.23.** *We have divisibilities*

$$\begin{aligned}\log^+(\gamma) & \mid L_{+\epsilon(p)p^{k-1}}(\phi^2) + L_{-\epsilon(p)p^{k-1}}(\phi^2); \\ \log^-(\gamma) & \mid L_{+\epsilon(p)p^{k-1}}(\phi^2) - L_{-\epsilon(p)p^{k-1}}(\phi^2).\end{aligned}$$

*Similarly,*

$$\begin{aligned}\log^+(\gamma) & \mid L_{+\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)) + L_{-\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)); \\ \log^-(\gamma) & \mid L_{+\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)) - L_{-\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)).\end{aligned}$$

*Proof.* The first set of divisibilities follows from Lemma 3.22. The second set is then immediate by definition.  $\square$

This allows us to define the following.

**Definition 3.24.** *We define the plus and minus  $p$ -adic  $L$ -functions for  $\text{Sym}^2(V_f)$  by*

$$\begin{aligned}L_p^+(\text{Sym}^2(V_f)) &= \left( L_{+\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)) + L_{-\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)) \right) / 2 \log^+(\gamma); \\ L_p^-(\text{Sym}^2(V_f)) &= \left( L_{+\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)) - L_{-\epsilon(p)p^{k-1}}(\text{Sym}^2(V_f)) \right) / 2 \log^-(\gamma).\end{aligned}$$

*Similarly, we define the plus and minus  $p$ -adic  $L$ -functions for  $V_2$  by*

$$\begin{aligned}L_p^+(\phi^2) &= \left( L_{+\epsilon(p)p^{k-1}}(\phi^2) + L_{-\epsilon(p)p^{k-1}}(\phi^2) \right) / 2 \log^+(\gamma); \\ L_p^-(\phi^2) &= \left( L_{+\epsilon(p)p^{k-1}}(\phi^2) - L_{-\epsilon(p)p^{k-1}}(\phi^2) \right) / 2 \log^-(\gamma).\end{aligned}$$

It is immediate that we have

$$(9) \quad L_p^\pm(\text{Sym}^2(V_f)) = L_p^\pm(\phi^2) \times \text{Tw}_{-k+1}(L_p(\varepsilon_K \cdot \epsilon)).$$

**Theorem 3.25.** *Both  $L_p^\pm(\text{Sym}^2(V_f))$  are elements of  $\Lambda_E(G_\infty)$  and admit the following interpolating properties:*

(a) *If  $n$  is even, then*

$$\begin{aligned}\chi^{2k-3} \theta \left( L_p^+(\text{Sym}^2(V_f)) \right) &= \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\log^+(\chi^{2k-3}\theta(\gamma)) \tau(\theta^{-1})^2 \epsilon(p)^n} \times \frac{L(\text{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1} \Omega_+}, \\ \chi^{2k-3} \left( L_p^+(\text{Sym}^2(V_f)) \right) &= \frac{(2k-3)!(k-1)!(1-p^{-1})}{\log^+(\chi^{2k-3}(\gamma))} \times \frac{L(\text{Sym}^2 f, 2k-2)}{(2\pi i)^{k-1} \Omega_+};\end{aligned}$$

(b) If  $n$  is odd, then

$$\begin{aligned}\chi^{2k-3}\theta\left(L_p^-(\mathrm{Sym}^2(V_f))\right) &= \frac{(2k-3)!(k-1)!p^{2n(k-1)}}{\log^-(\chi^{2k-3}\theta(\gamma))\tau(\theta^{-1})^2\epsilon(p)^n} \times \frac{L(\mathrm{Sym}^2 f, \theta^{-1}, 2k-2)}{(2\pi i)^{k-1}\Omega_+}, \\ \chi^{2k-3}\left(L_p^-(\mathrm{Sym}^2(V_f))\right) &= \frac{(2k-3)!(k-1)!(\epsilon(p)p^{-k+1} - \epsilon(p)^{-1}p^{k-2})}{\log^-(\chi^{2k-3}(\gamma))} \times \frac{L(\mathrm{Sym}^2 f, 2k-2)}{(2\pi i)^{k-1}\Omega_+}.\end{aligned}$$

Moreover,  $L_p^\pm(\mathrm{Sym}^2(V_f))$  are uniquely determined by (a) and (b) respectively.

*Proof.* By the first part of Theorem 3.20,  $L_{\pm\epsilon(p)p^{k-1}}(\mathrm{Sym}^2(V_f))$  are both elements of  $\mathcal{H}_{k-1,E}(G_\infty)$ . But  $\log^\pm(\gamma) \sim \log^{k-1}$ , so the quotients above are in  $\mathcal{H}_{0,E}(G_\infty) = \Lambda_E(G_\infty)$ .

The interpolating formulae in (a) and (b) follow from those given in Theorem 3.20.

Finally, since  $L_p^\pm(\mathrm{Sym}^2(V_f)) \in \Lambda_E(G_\infty)$ , they are uniquely determined by their values at an infinite number of characters, hence the last part of the theorem.  $\square$

**Lemma 3.26.** *Let  $\eta$  be an even character on  $\Delta$ , then  $L_p^{\pm,\eta}(\mathrm{Sym}^2(V_f)) \neq 0$ .*

*Proof.* The same as the proof of Lemma 3.21.  $\square$

**Remark 3.27.** Analogues of Theorem 3.25 and Lemma 3.26 for  $L_p^\pm(\phi^2)$  can be deduced in the same way.

**Remark 3.28.** A conjectural generalisation of Pollack's plus and minus splittings of  $p$ -adic  $L$ -functions for motives has been formulated in [2]. Theorem 3.25 gives an affirmative answer to Conjecture 2 of op. cit. for the special case when the motive corresponds to the symmetric square of a CM modular form.

#### 4. SELMER GROUPS

In this section, we define the plus and minus  $p$ -Selmer groups for  $\mathrm{Sym}^2(V_f)$  and relate them to the  $p$ -adic  $L$ -functions  $L_p^\pm(\mathrm{Sym}^2(V_f))$  defined above. By the decomposition given by Proposition 3.3, we only need to define their counterparts for  $V_2 = \tilde{V}_{\phi^2}$  because the Selmer group of  $V_1$  is relatively well-understood. The  $G_\mathbb{Q}$ -representation  $V_2$  behaves in exactly the same way as  $V_{f'}$  where  $f'$  is some CM modular form of weight  $2k-1$ , so many of the results on  $V_2$  below can be proved using the arguments given in [6]. Therefore, we only outline the proofs without giving all the details here.

**4.1. Coleman maps and Selmer groups.** As in [6, 7], we define plus and minus Selmer groups using the kernels of some Coleman maps.

**Proposition 4.1.** *If  $z \in \mathbb{H}_{\mathrm{Iw}}^1(V_2^*)$ , then*

$$\begin{aligned}\log^+(\gamma) & \mid \mathcal{L}_{\varphi(\omega) \otimes \varphi(\omega)}(z), \\ \log^-(\gamma) & \mid \mathcal{L}_{\omega \otimes \omega}(z).\end{aligned}$$

*Proof.* As in [6, Proposition 3.14], this can be proved using Proposition 3.10.  $\square$

Therefore, as in [6], we may define  $\Lambda_E(G_\infty)$ -homomorphisms

$$\begin{aligned}\mathrm{Col}^+ : \mathbb{H}_{\mathrm{Iw}}^1(V_2^*) & \rightarrow \Lambda_E(G_\infty) \\ z & \mapsto \frac{1}{2[\varphi(\omega) \otimes \varphi(\omega), \bar{\omega}] \log^+(\gamma)} \mathcal{L}_{\varphi(\omega) \otimes \varphi(\omega)}(z); \\ \mathrm{Col}^- : \mathbb{H}_{\mathrm{Iw}}^1(V_2^*) & \rightarrow \Lambda_E(G_\infty) \\ z & \mapsto \frac{1}{2[\varphi(\omega) \otimes \varphi(\omega), \bar{\omega}] \log^-(\gamma)} \mathcal{L}_{\omega \otimes \omega}(z).\end{aligned}$$

Then, it is clear by definition that  $\mathrm{Col}^\pm(\mathrm{Tw}_{2k-2}(z(\bar{\phi}^2))) = L_p^\pm(\phi^2)$ .

We now fix an  $\mathcal{O}_E$ -lattice  $T$  of  $V(\phi)$  which is stable under  $G_{\mathbb{Q}}$ , it then gives rise to natural  $\mathcal{O}_E$ -lattices  $T_f = \text{Ind}_K^{\mathbb{Q}}(T)$  and  $\text{Sym}^2 T_f$  in  $V_f = \hat{V}_{\phi}$  and  $\text{Sym}^2(V_f)$  respectively, both of which are again stable under  $G_{\mathbb{Q}}$ . As  $p \neq 2$ , we have

$$\text{Sym}^2 T_f \cong T_1 \oplus T_2 \quad \text{and} \quad \text{Sym}^2 V_f/T_f \cong V_1/T_1 \oplus V_2/T_2$$

for some  $\mathcal{O}_E$ -lattice  $T_i$  inside  $V_i$  for  $i = 1, 2$ .

Write  $H_{\pm}^1(\mathbb{Q}_{p,n}, T_2^*)$  for the projection of  $\ker(\text{Col}^{\pm})$  into  $H^1(\mathbb{Q}_{p,n}, T_2^*)$  and define  $H^1(\mathbb{Q}_{p,n}, V_2/T_2(1))^{\pm}$  to be the exact annihilator of  $H_{\pm}^1(\mathbb{Q}_{p,n}, T_2^*)$  under the Pontryagin duality

$$H^1(\mathbb{Q}_{p,n}, T_2^*) \times H^1(\mathbb{Q}_{p,n}, V_2/T_2(1)) \rightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

Let  $F$  be a number field. Then the  $p$ -Selmer group of  $\text{Sym}^2 T_f(1)$  decomposes into those of  $T_1(1)$  and  $T_2(1)$ :

$$\text{Sel}_p(\text{Sym}^2 T_f(1)/F) = \text{Sel}_p(T_1(1)/F) \oplus \text{Sel}_p(T_2(1)/F).$$

We define the plus/minus Selmer groups over  $k_n = \mathbb{Q}(\mu_{p^n})$  by

$$\begin{aligned} \text{Sel}_p^{\pm}(T_2(1)/k_n) &= \ker \left( \text{Sel}_p(T_2(1)/k_n) \rightarrow \frac{H^1(\mathbb{Q}_{p,n}, V_2/T_2(1))}{H_f^1(\mathbb{Q}_{p,n}, V_2/T_2(1))^{\pm}} \right), \\ \text{Sel}_p^{\pm}(\text{Sym}^2 T_f(1)/k_n) &= \text{Sel}_p(T_1(1)/k_n) \oplus \text{Sel}_p^{\pm}(T_2(1)/k_n) \end{aligned}$$

and let

$$\text{Sel}_p^{\pm}(T_2(1)/k_{\infty}) = \varinjlim \text{Sel}_p^{\pm}(T_2(1)/k_n) \quad \text{and} \quad \text{Sel}_p^{\pm}(\text{Sym}^2 T_f(1)/k_{\infty}) = \varinjlim \text{Sel}_p^{\pm}(\text{Sym}^2 T_f(1)/k_n).$$

**4.2. Description of the kernels.** In this section, we give a more explicit description of the groups  $H_f^1(\mathbb{Q}_{p,n}, V_2/T_2(1))^{\pm}$  under the following additional assumption.

**Hypothesis 4.2.** *Either  $p-1 \nmid k-1$  or  $\epsilon \neq 1$ .*

In [6, §4], one of the key ingredients to give an explicit description of  $H_f^1(\mathbb{Q}_{p,n}, V_f/T_f(1))^{\pm}$  is the fact that  $(V_f/T_f(j))^{G_{\mathbb{Q}_{p,n}}} = 0$  under some appropriate assumptions. We show below that we get an analogue of such description under Hypothesis 4.2.

**Lemma 4.3.** *If Hypothesis 4.2 holds, then  $(V_2/T_2(j))^{G_{\mathbb{Q}_{p,n}}} = 0$  for all  $j \in \mathbb{Z}$  and  $n \in \mathbb{Z}_{\geq 0}$ .*

*Proof.* Let  $q \nmid N$  be a prime which is inert in  $K$ . Then, by the second half of the proof of Proposition 3.3, we see that the eigenvalues of the  $q$ -Frobenius on  $V_2(j)$  are  $\pm \epsilon(q) \chi^j(q) q^{k-1}$ . Therefore, as in [6, proof of Lemma 4.4], it is enough to show that there exists some  $q$  such that

$$\pm \epsilon(q) \chi^j(q) q^{k-1} \not\equiv 1 \pmod{p}.$$

If either  $p-1 \nmid k-1$  or  $\epsilon(q) \neq 1$ , we can find such a  $q$  by Dirichlet's theorem, so we are done.  $\square$

**Corollary 4.4.** *If Hypothesis 4.2 holds, then the restriction map  $H^1(\mathbb{Q}_{p,m}, T_2(1)) \rightarrow H^1(\mathbb{Q}_{p,n}, T_2(1))$  is injective for any integers  $n \geq m \geq 0$ . On identifying the former as a subgroup of the latter, we have*

$$H_f^1(\mathbb{Q}_{p,n}, V_2/T_2(1))^{\pm} = H_f^1(\mathbb{Q}_{p,n}, T_2(1))^{\pm} \otimes E/\mathcal{O}_E.$$

Here

$$H_f^1(\mathbb{Q}_{p,n}, T_2(1))^{\pm} = \{x \in H_f^1(\mathbb{Q}_{p,n}, T_2(1)) : \text{cor}_{n/m+1}(x) \in H_f^1(\mathbb{Q}_{p,m}, T_2(1)) \forall m \in S_n^{\pm}\}$$

where  $\text{cor}$  denotes the corestriction map and

$$\begin{aligned} S_n^+ &= \{m \in [0, n-1] : m \text{ even}\}, \\ S_n^- &= \{m \in [0, n-1] : m \text{ odd}\}. \end{aligned}$$

*Proof.* These can be proved in exactly the same way as their counterparts in [6, §4] using Lemma 4.3.  $\square$

### 4.3. Main conjectures.

**Theorem 4.5.** *Let  $\theta$  be a character on  $\Delta$  and  $r \geq 0$  an integer such that  $\chi^{r+1}\theta(-1) = \eta(-1)$ . Then  $\text{Sel}_p(\mathbb{Z}_p(\eta)(r+1))^\theta$  is  $\Lambda_E(\Gamma)$ -cotorsion and*

$$\text{Char}_{\Lambda_E(\Gamma)} \left( \text{Sel}_p(\mathbb{Z}_p(\eta)(r+1))^{\vee, \theta} \right) = (\text{Tw}_{-r} L_p^\theta(\eta)).$$

*Proof.* For any  $\Lambda_E(G_\infty)$ -module,  $M^\vee(r) = M(-r)^\vee$ . If  $M$  is a  $\Lambda_E(\Gamma)$ -torsion module, we have  $\text{Char}(M(r)) = \text{Tw}_r(\text{Char}(M))$ . Therefore, the result is just a rewrite of the Iwasawa main conjecture, as proved by Mazur-Wiles [8].  $\square$

**Corollary 4.6.** *Let  $\eta$  be an even character on  $\Delta$ . Then*

$$\text{Char}_{\Lambda_E(\Gamma)} (\text{Sel}_p(T_1(1)/k_\infty)^{\vee, \eta}) = (\text{Tw}_{-k+1} L_p^\eta(\varepsilon_K \cdot \epsilon)).$$

*Proof.* We may apply Theorem 4.5 to  $\varepsilon_K \cdot \epsilon$  with  $r = k - 1$ .  $\square$

**Proposition 4.7.** *Let  $\delta = \pm$  and let  $\eta$  be a character on  $\Delta$  such that  $\eta = 1$  if  $\delta = -$ . Then,  $\text{Sel}_p^\delta(T_2(1)/k_\infty)^\theta$  is  $\Lambda_E(\Gamma)$ -cotorsion and*

$$\text{Char}_{\Lambda_E(\Gamma)} \left( \text{Sel}_p^\delta(T_2(1)/k_\infty)^{\vee, \eta} \right) = (L_p^{\delta, \eta}(\phi^2)).$$

*Proof.* This follows from the same argument as in [12], which has been generalised for CM modular forms in [6, §7]. It relies on the main conjecture for  $K$  as proved in [13].  $\square$

**Theorem 4.8.** *Let  $\eta$  be character on  $\Delta$  as in the statement of Proposition 4.7. Then  $\text{Sel}_p^\pm(\text{Sym}^2(V_f)/k_\infty)^\eta$  is  $\Lambda_E(\Gamma)$ -cotorsion and*

$$\text{Char}_{\Lambda_E(\Gamma)} (\text{Sel}_p^\pm(\text{Sym}^2(V_f)/k_\infty)^{\vee, \eta}) = (L_p^{\pm, \eta}(\text{Sym}^2(V_f))).$$

*Proof.* Recall that

$$\text{Sel}_p^\pm(\text{Sym}^2 T_f(1)/k_\infty) = \text{Sel}_p(T_1(1)/k_\infty) \oplus \text{Sel}_p^\pm(T_2(1)/k_\infty)$$

by definition, so

$$\text{Sel}_p^\pm(\text{Sym}^2 T_f(1)/k_\infty)^{\vee, \eta} = \text{Sel}_p(T_1(1)/k_\infty)^{\vee, \eta} \oplus \text{Sel}_p^\pm(T_2(1)/k_\infty)^{\vee, \eta}.$$

But we have

$$L_p^{\pm, \eta}(\text{Sym}^2(V_f)) = L_p^{\pm, \eta}(\phi^2) \times \text{Tw}_{-k+1}(L_p^\eta(\varepsilon_K \cdot \epsilon))$$

by (9). Therefore, the theorem follows from Corollary 4.6 and Proposition 4.7 because

$$\text{Char}(M_1 \oplus M_2) = \text{Char}(M_1) \text{Char}(M_2)$$

for any torsion modules  $M_1$  and  $M_2$ .  $\square$

## 5. APPENDIX

In this section, we fix an integer  $m \geq 2$ . We prove an analogue of Proposition 3.3.

**Proposition 5.1.** *If  $m$  is even, we have a decomposition of  $G_{\mathbb{Q}}$ -representations*

$$\text{Sym}^m V_f \cong \bigoplus_{i=0}^{m/2-1} \left( \tilde{V}_{\phi^{m-2i}} \otimes (\varepsilon_K \det \rho_f)^i \right) \oplus (\varepsilon_K \det \rho_f)^{m/2}.$$

*If  $m$  is odd, then*

$$\text{Sym}^m V_f \cong \bigoplus_{i=0}^{(m-1)/2} \left( \tilde{V}_{\phi^{m-2i}} \otimes (\varepsilon_K \det \rho_f)^i \right).$$

*Proof.* We only give the proof for the case when  $m$  is even since the other case can be proved in a similar way. Let  $x, y$  be the basis of  $V_f$  given as in §3.2. For an integer  $r$  such that  $0 \leq r \leq m$ , we write  $x_r$  for the element in  $V_f^{\otimes m}$  given by

$$\sum a_1 \otimes a_2 \otimes \cdots \otimes a_m$$

where the sum runs over  $a_i \in \{x, y\}$  with  $\#\{i : a_i = x\} = r$ . Then,  $x_0, \dots, x_m$  give a basis of  $\text{Sym}^m V_f$ .

If  $\sigma \in G_K$ , we have

$$\sigma(x_r) = \tilde{\phi}^r(\sigma) \tilde{\phi}^{m-r}(\iota\sigma\iota) x_r$$

by (2). If  $\sigma = \iota\sigma'$  with  $\sigma' \in G_K$ , then

$$\sigma(x_r) = \tilde{\phi}^r(\sigma') \tilde{\phi}^{m-r}(\iota\sigma'\iota) x_{m-r}$$

by (3). Therefore,  $x_r$  and  $x_{m-r}$  generate a subrepresentation of  $\text{Sym}^m V_f$ , which we denote by  $\rho_r : G_{\mathbb{Q}} \rightarrow \text{GL}(V_r)$  where  $0 \leq r \leq m/2$ . Note that  $V_r$  is 2-dimensional if  $r < m/2$  and  $V_{m/2}$  is 1-dimensional. We have a decomposition

$$\text{Sym}^m V_f \cong \bigoplus_{r=0}^{m/2} V_r.$$

For  $r < m/2$ , the matrix of  $\sigma \in G_K$  respect to the basis  $x_{m-r}, x_r$  is

$$\begin{pmatrix} \tilde{\phi}^{m-r}(\sigma) \tilde{\phi}^r(\iota\sigma\iota) & 0 \\ 0 & \tilde{\phi}^r(\sigma) \tilde{\phi}^{m-r}(\iota\sigma\iota) \end{pmatrix} = \tilde{\phi}^r(\sigma\iota\sigma\iota) \begin{pmatrix} \tilde{\phi}^{m-2r}(\sigma) & 0 \\ 0 & \tilde{\phi}^{m-2r}(\iota\sigma\iota) \end{pmatrix},$$

whereas that of  $\sigma = \iota\sigma'$  with  $\sigma' \in G_K$  is given by

$$\begin{pmatrix} 0 & \tilde{\phi}^r(\sigma') \tilde{\phi}^{m-r}(\iota\sigma'\iota) \\ \tilde{\phi}^{m-r}(\sigma') \tilde{\phi}^r(\iota\sigma'\iota) & 0 \end{pmatrix} = \tilde{\phi}^r(\sigma' \iota \sigma' \iota) \begin{pmatrix} 0 & \tilde{\phi}^{m-2r}(\iota\sigma'\iota) \\ \tilde{\phi}^{m-2r}(\sigma') & 0 \end{pmatrix}.$$

Therefore, we see that  $\rho_r \cong \text{Ind}_K^{\mathbb{Q}}(V(\phi^{m-2r})) \cdot (\varepsilon_K \det \rho_f)^r$  by Lemma 3.2.

Finally, for  $r = m/2$ , we have

$$\sigma(x_{m/2}) = \begin{cases} \tilde{\phi}^{m/2}(\sigma\iota\sigma\iota) x_{m/2} & \text{if } \sigma \in G_K \\ \tilde{\phi}^{m/2}(\sigma' \iota \sigma' \iota) x_{m/2} & \text{if } \sigma = \iota\sigma' \text{ where } \sigma' \in G_K. \end{cases}$$

Hence,  $V_{m/2} = (\varepsilon_K \det \rho_f)^{m/2}$  again by Lemma 3.2. This finishes the proof.  $\square$

**Corollary 5.2.** *The complex L-function admits a factorisation*

$$L(\text{Sym}^m f, s) = \begin{cases} \left( \prod_{i=0}^{m/2-1} L(\phi^{m-2i}, (\varepsilon_K \epsilon)^i, s - i(k-1)) \right) L((\varepsilon_K \epsilon)^{m/2}, s - m/2(k-1)) & \text{if } m \text{ is even,} \\ \prod_{i=0}^{(m-1)/2} L(\phi^{m-2i}, (\varepsilon_K \epsilon)^i, s - i(k-1)) & \text{otherwise.} \end{cases}$$

*Proof.* This can be proved in the same way as Corollary 3.4.  $\square$

**Remark 5.3.** *For  $0 \leq i \leq \lfloor (m-1)/2 \rfloor$ , we may obtain a p-adic L-function that interpolates the L-values of  $\phi^{m-2i}$  at  $(m-2i)(k-1)$  using Proposition 3.1. However, when  $m > 2$ , their product does not interpolate the L-values of  $\text{Sym}^m f$ . We would need p-adic L-functions that interpolate the L-values of  $\phi^{m-2i}$  at  $(m-i)(k-1)$  instead.*

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